the high wave numbers, δ is taken as small parameter. Let $\mathbf{u}^\delta$ be the solution to the problem:

$$\frac{\partial u_i^\delta}{\partial t} + \frac{\partial (u_i^\delta + v_i^\delta)(u_j^\delta + v_j^\delta)}{\partial x_j} - \nu \frac{\partial^2 u_i^\delta}{\partial x_k \partial x_k} = -\frac{\partial p^\delta}{\partial x_i} - \frac{\partial v_i^\delta}{\partial t} + \nu \frac{\partial^2 v_i^\delta}{\partial x_k \partial x_k} \quad . \tag{6.55}$$

If $\mathbf{v}^\delta$ is close to $\mathbf{u}'$, then $\overline{\mathbf{u}^\delta}$ is close to $\overline{\mathbf{u}}$. More precisely, we have:

$$\mathbf{u}^\delta = \overline{\mathbf{u}} + \delta \mathbf{u}^1 + \delta^2 \mathbf{u}^2 + ... \tag{6.56}$$

A modeling of this kind, while satisfactory on the theoretical level, is not so in practice because the function $\mathbf{v}^\delta$ oscillates very quickly in space and time, and the number of degrees of freedom needed in the discrete system to describe its variations remains very high. To reduce the size of the discrete system significantly, other hypotheses are needed, leading to the definition of simplified models which are described in the following.

First Model. The first simplification consists in choosing the random process in the form:

$$\mathbf{v}^\delta(\mathbf{x}, t) = \frac{1}{\delta} \mathbf{v}\left(\mathbf{x}, t, \mathbf{x}', t'\right) \quad , \tag{6.57}$$

in the space and time scales $\mathbf{x}'$ and t', respectively, of the subgrid modes are defined as:

$$\mathbf{x}' = \frac{\mathbf{x} - \overline{\mathbf{u}} t}{\delta}, \; t' = \frac{t}{\delta^2} \quad . \tag{6.58}$$

The new variable $\mathbf{v}\left(\mathbf{x}, t, \mathbf{x}', t'\right)$ oscillates slowly and can thus be represented with fewer degrees of freedom. Assuming that $\mathbf{v}$ is periodical depending on the variables $\mathbf{x}'$ and t' on a domain $\Omega_v = Z \times]0, T'[$, and that the average of $\mathbf{v}$ is null over this domain[6], it is demonstrated that the subgrid tensor is expressed in the form:

$$\tau = B \nabla \overline{\mathbf{u}} \quad , \tag{6.59}$$

where the term $B\nabla\overline{\mathbf{u}}$ is computed by taking the average on the cell of periodicity Ω_v of the term $(\mathbf{v} \cdot \nabla \mathbf{u}^1 + \mathbf{u}^1 \cdot \nabla \mathbf{v})$, where $\mathbf{u}^1$ is the a solution on this cell of the problem:

$$\frac{\partial \mathbf{u}^1}{\partial t'} - \nu \nabla^2_{\mathbf{x}'} \mathbf{u}^1 + \mathbf{v} \cdot \nabla_{\mathbf{x}'} \mathbf{u}^1 + \mathbf{u}^1 \cdot \nabla_{\mathbf{x}'} \mathbf{v} = \nabla q - \mathbf{v} \cdot \nabla \overline{\mathbf{u}} - \overline{\mathbf{u}} \cdot \nabla \mathbf{v} \quad , \tag{6.60}$$

6 This is equivalent to considering that $\mathbf{v}\left(\mathbf{x}, t, \mathbf{x}', t'\right)$ is statistically homogeneous and isotropic, which is theoretically justifiable by the physical hypothesis of local isotropy.

$$\nabla_{\mathbf{x}'} \cdot \mathbf{u}^1 = 0 \quad , \tag{6.61}$$

where $\nabla_{\mathbf{x}'}$ designates the gradient with respect to the $\mathbf{x}'$ variables and q the Lagrange multiplier that enforces the constraint (6.61). This model, though simpler, is still difficult to use because the variable $(\mathbf{x} - \overline{\mathbf{u}}t)$ is difficult to manipulate. So other simplifications are needed.

Second Model. To arrive at a usable model, the authors propose neglecting the transport of the random variable by the filtered field in the field's evolution equation. This way, the random variable can be chosen in the form:

$$\mathbf{v}^\delta(\mathbf{x}, t) = \frac{1}{\delta} \mathbf{v}\left(\mathbf{x}, t, \mathbf{x}'', t'\right) \quad , \tag{6.62}$$

with

$$\mathbf{x}'' = \frac{\mathbf{x}}{\delta} \quad , \tag{6.63}$$

and where the time t' is defined as before. Assuming that $\mathbf{v}$ is periodic along $\mathbf{x}''$ and t' on the domain Ω_v and has an average of zero over this interval, the subgrid term takes the form:

$$\tau = A \nabla \overline{\mathbf{u}} \quad , \tag{6.64}$$

where A is a definite positive tensor such that the term $A\nabla\overline{\mathbf{u}}$ is equal to the average of the term $(\mathbf{v} \otimes \mathbf{u}^1)$ over Ω_v, in which $\mathbf{u}^1$ is a solution on Ω_v of the problem:

$$\frac{\partial \mathbf{u}^1}{\partial t'} - \nu \nabla^2_{\mathbf{x}''} \mathbf{u}^1 + \mathbf{v} \cdot \nabla_{\mathbf{x}''} \mathbf{u}^1 = \nabla q + \mathbf{v} \cdot \nabla \overline{\mathbf{u}} \quad , \tag{6.65}$$

$$\nabla_{\mathbf{x}''} \cdot \mathbf{u}^1 = 0 \quad . \tag{6.66}$$

6.2 Differential Subgrid Stress Models

6.2.1 Deardorff Model

Another approach for obtaining a model for the subgrid tensor consists in solving an evolution equation for each of its components. This approach, proposed by Deardorff [77] and recently re-investigated by Fureby *et al.* [107], is analogous in form to two-point statistical modeling. Here, we adopt the case where the filter is a Reynolds operator. The subgrid tensor τ_{ij} is thus reduced to the subgrid Reynolds tensor R_{ij}. We deduce the evolution equation of the subgrid tensor components from that of the subgrid modes (3.29)[7]:

7 This is done by applying the filter to the relation obtained by multiplying (3.29) by u'_j and taking the half-sum with the relation obtained by inverting the subscripts i and j.

$$\begin{aligned}\frac{\partial \tau_{ij}}{\partial t} &= -\frac{\partial}{\partial x_k}(\overline{u}_k \tau_{ij}) - \tau_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{u}_i}{\partial x_k} \\ &\quad -\frac{\partial}{\partial x_k}\overline{u'_i u'_j u'_k} + \overline{p'\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)} \\ &\quad -\frac{\partial}{\partial x_j}\overline{u'_i p'} - \frac{\partial}{\partial x_i}\overline{u'_j p'} - 2\nu \overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} \quad . \end{aligned} \tag{6.67}$$

The various terms in this equation have to be modeled. The models Deardorff proposes are:

– For the pressure–strain correlation term:

$$\overline{p'\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)} = -C_{\mathrm{m}}\frac{\sqrt{q^2_{\mathrm{sgs}}}}{\overline{\Delta}}\left(\tau_{ij} - \frac{2}{3}q^2_{\mathrm{sgs}}\delta_{ij}\right) + \frac{2}{5}q^2_{\mathrm{sgs}}\overline{S}_{ij} \quad , \tag{6.68}$$

where C_{m} is a constant, q^2_{sgs} the subgrid kinetic energy, and $\overline{S}_{ij}$ the strain rate tensor of the resolved field.

– For the dissipation term:

$$\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} = \delta_{ij}C_{\mathrm{e}}\frac{(q^2_{\mathrm{sgs}})^{3/2}}{\overline{\Delta}} \quad , \tag{6.69}$$

where C_{e} is a constant.

– For the triple correlations:

$$\overline{u'_i u'_j u'_k} = -C_{3\mathrm{m}}\overline{\Delta}\left(\frac{\partial}{\partial x_i}\tau_{jk} + \frac{\partial}{\partial x_j}\tau_{ik} + \frac{\partial}{\partial x_k}\tau_{ij}\right) \quad . \tag{6.70}$$

The pressure–velocity correlation terms $\overline{p'u'_i}$ are neglected. The values of the constants are determined in the case of isotropic homogeneous turbulence:

$$C_{\mathrm{m}} = 4.13, \; C_{\mathrm{e}} = 0.70, \; C_{3\mathrm{m}} = 0.2 \quad . \tag{6.71}$$

Lastly, the subgrid kinetic energy is determined using evolution equation (4.108).

6.2.2 Link with the Subgrid Viscosity Models

We reach the functional subgrid viscosity models again starting with a model with transport equations for the subgrid stresses, at the cost of additional assumptions. For example, Yoshizawa *et al.* [363] proposed neglecting all the terms of equation (6.67), except those of production. The evolution equation thus reduced comes to:

$$\frac{\partial \tau_{ij}}{\partial t} = -\tau_{ik}\frac{\partial \overline{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{u}_i}{\partial x_k} \quad . \tag{6.72}$$

Assuming that the subgrid modes are isotropic or quasi-isotropic, *i.e.* that the extra-diagonal elements of the subgrid tensor are very small compared with the diagonal elements, and that the latter are almost mutually equal, the right-hand side of the reduced equation (6.72) comes down to the simplified form:

$$-q^2_{\mathrm{sgs}}\overline{S}_{ij} \quad , \tag{6.73}$$

in which $q^2_{\mathrm{sgs}} = \overline{u'_k u'_k}/2$ is the subgrid kinetic energy. Let t_0 be the characteristic time of the subgrid modes. Considering the relations (6.72) and (6.73), and assuming that the relaxation time of the subgrid modes is much shorter than that of the resolved scales[8], we get

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} \approx -t_0 q^2_{\mathrm{sgs}}\overline{S}_{ij} \quad . \tag{6.74}$$

The time t_0 can be evaluated by dimensional argument using the cutoff length $\overline{\Delta}$ and the subgrid kinetic energy:

$$t_0 \approx \frac{\overline{\Delta}}{\sqrt{q^2_{\mathrm{sgs}}}} \quad . \tag{6.75}$$

By entering this estimate into equation (6.74), we get an expression analogous to the one used in the functional modeling framework:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} \approx -\overline{\Delta}\sqrt{q^2_{\mathrm{sgs}}}\,\overline{S}_{ij} \quad . \tag{6.76}$$

6.3 Deterministic Models of the Subgrid Structures

6.3.1 General

Misra and Pulin [236], following on the works of Pullin and Saffman [273], proposed subgrid models using the assumption that the subgrid modes can be represented by stretched vortices whose orientation is governed by the resolved scales.

Supposing that the subgrid modes can be linked to a random superimposition of fields generated by axisymmetrical vortices, the subgrid tensor can be written in the form:

$$\tau_{ij} = 2\int_{k_c}^{\infty} E(k)dk\langle E_{pi}Z_{pq}E_{qj}\rangle \quad , \tag{6.77}$$

[8] We again find here the [illegible]al scale-separation hypothesis 4.4.